SEMISOLVABILITY OF SEMISIMPLE HOPF ALGEBRAS OF DIMENSION $2q^3$

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Abstract. Let q be a prime number, k an algebraically closed field of characteristic 0, and H a semisimple Hopf algebra of dimension $2q^3$. This paper proves that H is always semisolvable.

1. Introduction

The notions of upper and lower semisolvability for finite-dimensional Hopf algebras were introduced by Montgomery and Whiterspoon [16], as generalizations of the notion of solvability for finite groups. In particular, if a finite-dimensional Hopf algebra A is semisolvable then A can be obtained by a number of extensions from group algebras or duals of group algebras. Therefore, in analogy with the situations for finite groups, it is enough for many applications to know that a Hopf algebra is semisolvable.

The known examples of semisimple Hopf algebras which are semisolvable are those of dimension p^n, pq^2, pqr and those of dimension less than 60, where p, q, r are distinct prime numbers and n is a natural number. See [16, 6, 20] for details. Our present work is devoted to providing a new class of semisimple Hopf algebras which are semisolvable. It can also be viewed as a generalization of [20, Chapter 12].

The paper is organized as follows. In Section 2, we recall the definitions and some of the basic properties of semisolvability, characters, Radford's biproducts and Drinfeld double, respectively. Some useful lemmas are also contained in this section.

In Section 3, we present our main results. Let $G(H)$ denote the group of group-like elements in a semisimple Hopf algebra H. By examining every possible order of $G(H)$, we prove that if the dimension of H is $2q^3$ then H is semisolvable, where q is a prime number. Moreover, if $|G(H)| = 2$ then H is group theoretical in the sense of [18].

Throughout this paper, all modules and comodules are left modules and left comodules, and moreover they are finite-dimensional over an algebraically closed field k of characteristic 0. $\otimes$, dim mean $\otimes_k$, $\dim_k$, respectively. If G is a finite group, kG denotes the group algebra of G, and k^G means $(kG)^*$. If $g \in G$ then $\langle g \rangle$ denotes the subgroup of G generated by g. Further C_n denotes the cyclic group of order n. For two positive integers m and n, $gcd(m, n)$ denotes the greatest common divisor of m, n. Our references for the theory of Hopf algebras are [17] or [28].

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2. Preliminaries

2.1. **Characters.** Throughout this subsection, H will be a semisimple Hopf algebra over k. The main result in [11] states that H is also cosemisimple.

We next recall some of the terminology and conventions from [22] that will be used throughout this paper.

Let V be an H-comodule. The character of V is the element $\chi = \chi_V \in H$ defined by $\langle f, \chi \rangle = \mathrm{Tr}_V(f)$ for all $f \in H^*$. The degree of χ is defined to be the integer $\mathrm{deg}\chi = \varepsilon(\chi) = \mathrm{dim}V$. We shall use X_t to denote the set of all irreducible characters of H of degree t.

All irreducible characters of H span a subalgebra $R(H^*)$ of H, which is called the character algebra of H^*. The antipode S induces an anti-algebra involution $* : R(H^*) \to R(H^*)$, given by $\chi \mapsto \chi^* := S(\chi)$.

Let $\chi_U, \chi_V \in R(H^*)$ be the characters of the H-comodules U and V, respectively. The integer $m(\chi_U, \chi_V) = \mathrm{dimHom}^H(U, V)$ is defined to be the multiplicity of U in V. Let $\widehat{H}$ denote the set of irreducible characters of H. Then $\widehat{H}$ is a basis of $R(H^*)$. If $\chi \in R(H^*)$, we may write $\chi = \sum_{\alpha \in \widehat{H}} m(\alpha, \chi)\alpha$.

For any group-like element g in $G(H)$, $m(g, \chi\chi^*) > 0$ if and only if $m(g, \chi\chi^*) = 1$ if and only if $g\chi = \chi$. The set of such group-like elements forms a subgroup of $G(H)$, of order at most $(\mathrm{deg}\chi)^2$. See [22, Theorem 10]. Denote this subgroup by $G[\chi]$. In particular, we have

$$\chi\chi^* = \sum_{g \in G[\chi]} g + \sum_{\alpha \in \widehat{H}, \mathrm{deg}\alpha > 1} m(\alpha, \chi\chi^*)\alpha. \tag{2.1}$$

A subalgebra A of $R(H^*)$ is called a standard subalgebra if A is spanned by irreducible characters of H. Let X be a subset of $\widehat{H}$. Then X spans a standard subalgebra of $R(H^*)$ if and only if the product of characters in X decomposes as a sum of characters in X. There is a bijection between $*$-invariant standard subalgebras of $R(H^*)$ and Hopf subalgebras of H. See [22, Theorem 6].

H is said to be of type $(d_1, n_1; \cdots; d_s, n_s)$ as a coalgebra if $d_1 = 1, d_2, \cdots, d_s$ are the dimensions of the irreducible H-comodules and n_i is the number of the non-isomorphic irreducible H-comodules of dimension d_i. That is, as a coalgebra, H is isomorphic to a direct sum of full matrix coalgebras

$$H \cong k^{(n_1)} \oplus \bigoplus_{i=2}^{s} M_{d_i}(k)^{(n_i)}. \tag{2.2}$$

If H^* is of type $(d_1, n_1; \cdots; d_s, n_s)$ as a coalgebra, then H is said to be of type $(d_1, n_1; \cdots; d_s, n_s)$ as an algebra.

Lemma 2.1. *Let χ be an irreducible character of H. Then*

(1) The order of $G[\chi]$ divides $(\mathrm{deg}\chi)^2$.

(2) The order of $G(H)$ divides $n(\mathrm{deg}\chi)^2$, where n is the number of non-isomorphic irreducible characters of degree $\mathrm{deg}\chi$.

Proof. It follows from Nichols-Zoeller Theorem [23]. See also [21, Lemma 2.2.2]. □

2.2. **Semisolvability.** Let B be a finite-dimensional Hopf algebra over k. A Hopf subalgebra $A \subseteq B$ is called normal if $h_1 A S(h_2) \subseteq A$ and $S(h_1)Ah_2 \subseteq A$, for all $h \in B$. If B does not contain proper normal Hopf subalgebras then it is called simple. Dualizing the notion of normal Hopf subalgebra, we obtain the notion of

conormal quotient Hopf algebra. The notion of simplicity is self-dual, that is, B is simple if and only if B^* is simple.

Let $K \subseteq A$ be a normal Hopf subalgebra. Then $B = A/AK^+$ is a conormal quotient Hopf algebra and the sequence of Hopf algebra maps $k \to K \to A \to B \to k$ is an exact sequence of Hopf algebras. In this case we shall say that A is an extension of B by K.

The extension above is called abelian if K is commutative and B is cocommutative. In this case $K \cong k^N$ and $B \cong kF$, for some finite groups N and F.

The following lemma is a direct consequence of [20, Corollary 1.4.3].

Lemma 2.2. *Let $\pi : A \to B$ be a conormal quotient Hopf algebra. Suppose that* $\dim B$ *is the least prime number dividing* $\dim A$. *Then $G(B^*) \subseteq Z(A^*) \cap G(A^*)$.*

Let $\pi : H \to B$ be a Hopf algebra map and consider the subspaces of coinvariants

$$H^{co\pi} = \{h \in H | (id \otimes \pi)\Delta(h) = h \otimes 1\}, \text{and}$$
$${}^{co\pi}H = \{h \in H | (\pi \otimes id)\Delta(h) = 1 \otimes h\}.$$

Then $H^{co\pi}$ (respectively, ${}^{co\pi}H$) is a left (respectively, right) coideal subalgebra of H. Moreover, we have

$$\dim H = \dim H^{co\pi} \dim \pi(H) = \dim {}^{co\pi}H \dim \pi(H).$$

The left coideal subalgebra $H^{co\pi}$ is stable under the left adjoint action of H. Moreover $H^{co\pi} = {}^{co\pi}H$ if and only if $H^{co\pi}$ is a (normal) Hopf subalgebra of H. If this is the case, we shall say that the map $\pi : H \to B$ is normal. See [26] for more details.

The following lemma comes from [20, Section 1.3].

Lemma 2.3. *Let $\pi : H \to B$ be a Hopf surjection and A a Hopf subalgebra of H such that $A \subseteq H^{co\pi}$. Then* $\dim A$ *divides* $\dim H^{co\pi}$.

By definition, H is called lower semisolvable if there exists a chain of Hopf subalgebras

$$H_{n+1} = k \subseteq H_n \subseteq \cdots \subseteq H_1 = H$$

such that H_{i+1} is a normal Hopf subalgebra of H_i, for all i, and all quotients $H_i/H_iH_{i+1}^+$ are trivial. That is, they are isomorphic to a group algebra or a dual group algebra. Dually, H is called upper semisolvable if there exists a chain of quotient Hopf algebras

$$H_{(0)} = H \xrightarrow{\pi_1} H_{(1)} \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_n} H(n) = k$$

such that $H_{(i-1)}^{co\pi_i}$ is a normal Hopf subalgebra of $H_{(i-1)}$, and all $H_{(i-1)}^{co\pi_i}$ are trivial.

By [16, Corollary 3.3], we have that H is upper semisolvable if and only if H^* is lower semisolvable.

Proposition 2.4. *Let H be a semisimple Hopf algebra of dimension pq^3, where p, q are distinct prime numbers. If H is not simple as a Hopf algebra then it is semisolvable.*

Proof. By assumption, H has a proper normal Hopf subalgebra K. Moreover, by Nichols-Zoeller Theorem [23], $\dim K$ divides $\dim H = pq^3$. We shall examine every possible $\dim K$.

If $\dim K = q^2$ or pq then $k \subseteq K \subseteq H$ is a chain such that K and H/HK^+ are both trivial (see [5, 13]). Hence, H is lower semisolvable.

If $\dim K = q^3$ then [13] shows that K has a non-trivial central group-like element g. Let $L = k\langle g\rangle$ be the group algebra of the cyclic group $\langle g\rangle$ generated by g. Then $k \subseteq L \subseteq K \subseteq H$ is a chain such that $L, K/KL^+$ and H/HK^+ are all trivial (see [29]). Hence, H is lower semisolvable.

If $\dim K = pq^2$ then [4, Lemma 2.2] and [19, Theorem 5.4.1] show that K has a proper normal Hopf subalgebra L of dimension p, q, pq or q^2. Then $k \subseteq L \subseteq K \subseteq H$ is a chain such that $L, K/KL^+$ and H/HK^+ are all trivial. Hence, H is lower semisolvable.

Finally, we consider the case that $\dim K = p$ or q. Let L be a proper normal Hopf subalgebra of H/HK^+ (Notice that H/HK^+ is not simple). Write $\overline{K} = H/HK^+$ and $\overline{L} = \overline{K}/\overline{K}L^+$. Then $H \xrightarrow{\pi_1} \overline{K} \xrightarrow{\pi_2} \overline{L} \to k$ is a chain such that every map is normal and $H^{co\pi_1}$, $(\overline{K})^{co\pi_2}$ are trivial. Hence, H is upper semisolvable. □

Remark 2.5. Let H be a semisimple Hopf algebra of dimension p^2q^2, where p, q are prime numbers. If H is not simple then H is also semisolvable. Indeed, since every (quotient) Hopf subalgebra of H is semisolvable, a similar argument as in Proposition 2.4 will prove the claim.

2.3. **Drinfeld double.** For a semisimple Hopf algebra H, $D(H) = H^{*cop} \bowtie H$ will denote the Drinfeld double of H. $D(H)$ is a Hopf algebra with underlying vector space $H^{*cop} \otimes H$. The main result in [7] proves that if V is an irreducible module of $D(H)$, then the dimension of V divides the dimension of H.

Let ${}^H_H\mathcal{YD}$ denote the category of (left-left) Yetter-Drinfeld modules over H. Objects of this category are vector spaces V endowed with an H-coaction $\rho : V \to H \otimes V$ and an H-action $\cdot : H \otimes V \to V$, which satisfies the compatibility condition $\rho(h \cdot v) = h_1 v_{-1} S(h_3) \otimes h_2 \cdot v_0$, for all $v \in V, h \in H$. Morphisms of this category are H-linear and colinear maps.

Majid first proved that the Yetter-Drinfeld category ${}^H_H\mathcal{YD}$ can be identified with the category ${}_{D(H)}\mathcal{M}$ of left modules over the quantum double $D(H)$. (see [12, Proposition 2.1]).

More details on $D(H)$ can be found in [17, Section 10.3]. The following theorem follows directly from [25, Proposition 9,10].

Theorem 2.6. *Suppose that H is a semisimple Hopf algebra.*

(1) The map $G(H^*) \times G(H) \to G(D(H))$, given by $(\eta, g) \mapsto \eta \bowtie g$, is a group isomorphism.

(2) Every group-like element of $D(H)^*$ is of the form $g \otimes \eta$, where $g \in G(H)$ and $\eta \in G(H^*)$. Moreover, $g \otimes \eta \in G(D(H)^*)$ if and only if $\eta \bowtie g$ is in the center of $D(H)$.

Corollary 2.7. *Suppose that H is a finite-dimensional Hopf algebra such that $G(D(H)^*)$ is non-trivial. If $gcd(|G(H)|, |G(H^*)|) = 1$ then H or H^* has a non-trivial central group-like element.*

Proof. Let $1 \neq g \otimes \eta \in G(D(H)^*)$. We may assume that $1 \neq g \in G(H)$, since otherwise $\eta \in G(H^*)$ would be a non-trivial central group-like element, and similarly we may assume that $\varepsilon \neq \eta \in G(H^*)$. Since $gcd(|G(H)|, |G(H^*)|) = 1$, the order of g and η are different. Assume that the order of g is n. Then $(g \otimes \eta)^n = g^n \otimes \eta^n = 1 \otimes \eta^n \neq 1 \otimes \varepsilon$ implies that $\eta^n \bowtie 1$ is in the center of $D(H)$. Hence, η^n is a non-trivial central group-like element in $G(H^*)$. Similarly, we can prove that $G(H)$ also has a non-trivial central group-like element. □

Recall that a semisimple Hopf algebra A is called of Frobenius type if the dimensions of the simple A-modules divide the dimension of A. Kaplansky conjectured that every finite-dimensional semisimple Hopf algebra is of Frobenius type [9, Appendix 2]. It is still an open problem. However, many examples show that a positive answer to Kaplansky's conjecture would be very helpful in the classification of semisimple Hopf algebras.

Lemma 2.8. *Suppose that H is a semisimple Hopf algebra of dimension pq^3, where p, q are prime numbers such that $p^2 < q$. Then the order of $G(D(H)^*)$ can not be q.*

Proof. Suppose on the contrary that $|G(D(H)^*)| = q$. Recall that $D(H)$ is of Frobenius type (see [4, Lemma 2.2] or [7]) and the dimension of every irreducible $D(H)$-module divides $\dim H$. Then we have an equation by applying (2.2) to $D(H)$

$$p^2q^6 = q + p^2a_1 + p^2q^2a_2 + p^2q^4a_3 + q^2a_4 + q^4a_5 + q^6a_6,$$

where $a_i (i = 1, \cdots, 6)$ are non-negative integers. Obviously, $a_1 \neq 0$, since otherwise the equation above can not hold. Let χ be an irreducible character of $D(H)$ of degree p. Then the decomposition of $\chi\chi^*$ as (2.1) gives rise to a contradiction, by Lemma 2.1 and the fact that $p^2 < q$. □

Let $g \in G(H)$, $\eta \in G(H^*)$, and $V_{g,\eta}$ denote the one-dimensional vector space endowed with the action $h \cdot 1 = \eta(h)1$, for all $h \in H$, and the coaction $1 \mapsto g \otimes 1$.

Lemma 2.9. [20, Lemma 1.6.1] *The one-dimensional Yetter-Drinfeld modules of H are exactly of the form $V_{g,\eta}$, where $g \in G(H)$ and $\eta \in G(H^*)$ are such that $(\eta \rightharpoonup h)g = g(h \leftharpoonup \eta)$ for all $h \in H$, where $\rightharpoonup$ and $\leftharpoonup$ are the regular actions of H^* on H.*

2.4. **Radford's biproduct.** Let A be a semisimple Hopf algebra and let ${}^A_A\mathcal{YD}$ denote the braided category of Yetter-Drinfeld modules over A. Let R be a semisimple Yetter-Drinfeld Hopf algebra in ${}^A_A\mathcal{YD}$. Denote by $\rho : R \to A \otimes R$, $\rho(a) = a_{-1} \otimes a_0$, and $\cdot : A \otimes R \to R$, the coaction and action of A on R, respectively. We shall use the notation $\Delta(a) = a^1 \otimes a^2$ and S_R for the comultiplication and the antipode of R, respectively.

Since R is in particular a module algebra over A, we can form the smash product (see [17, Definition 4.1.3]). This is an algebra with underlying vector space $R \otimes A$, multiplication is given by

$$(a \otimes g)(b \otimes h) = a(g_1 \cdot b) \otimes g_2h, \text{ for all } g, h \in A, a, b \in R,$$

and unit $1 = 1_R \otimes 1_A$.

Since R is also a comodule coalgebra over A, we can dually form the smash coproduct. This is a coalgebra with underlying vector space $R \otimes A$, comultiplication is given by

$$\Delta(a \otimes g) = a^1 \otimes (a^2)_{-1}g_1 \otimes (a^2)_0 \otimes g_2, \text{ for all } h \in A, a \in R,$$

and counit $\varepsilon_R \otimes \varepsilon_A$.

As observed by D. E. Radford (see [24, Theorem 1]), the Yetter-Drinfeld condition assures that $R \otimes A$ becomes a Hopf algebra with these structures. This Hopf algebra is called the Radford's biproduct of R and A. We denote this Hopf algebra by $R\#A$ and write $a\#g = a \otimes g$ for all $g \in A, a \in R$. Its antipode is given by

$$S(a\#g) = (1\#S(a_{-1}g))(S_R(a_0)\#1), \text{ for all } g \in A, a \in R.$$

A biproduct $R\#A$ as described above is characterized by the following property(see [24, Theorem 3]): suppose that H is a finite-dimensional Hopf algebra endowed with Hopf algebra maps $\iota : A \to H$ and $\pi : H \to A$ such that $\pi\iota : A \to A$ is an isomorphism. Then the subalgebra $R = H^{co\pi}$ has a natural structure of Yetter-Drinfeld Hopf algebra over A such that the multiplication map $R\#A \to H$ induces an isomorphism of Hopf algebras.

Following [27, Proposition 1.6], $H \cong R\#A$ is a biproduct if and only if $H^* \cong R^*\#A^*$ is a biproduct.

R is called trivial if R is an ordinary Hopf algebra. In particular, R is an ordinary Hopf algebra if A is normal in H, since $R \cong H/HA^+$ as a coalgebra.

The following lemma is a special case of [20, Lemma 4.1.9].

Lemma 2.10. *Let H be a semisimple Hopf algebra of dimension pq^3, where p, q are distinct prime numbers. If p divides both $|G(H)|$ and $|G(H^*)|$, then $H \cong R\#kG$ is a biproduct, where kG is the group algebra of group G of order p, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^3.*

3. Semisimple Hopf algebras of dimension $2q^3$

In this section, H will be a non-trivial semisimple Hopf algebra of dimension $2q^3$, where q is a prime number. Our main aim is to prove that H is semisolvable. By Proposition 2.4, it suffices to prove that H is not simple. When $q = 2$, the result follows from [16, Theorem 3.5]. When $q = 3$, the result has been obtained in [20, Chapter 12]. Therefore, in the rest of this section, we always assume that $q \geq 5$.

By [4, Lemma 2.2], H is of Frobenius type and $|G(H)| \neq 1$. Therefore, the dimension of a simple H-module can only be $1, 2, q$ or $2q$. It follows that we have an equation

$$2q^3 = |G(H)| + 4a + q^2 b + 4q^2 c, \tag{3.1}$$

where a, b, c are the numbers of non-isomorphic simple H-modules of dimension $2, q$ and $2q$, respectively. By Nichols-Zoeller Theorem [23], the order of $G(H)$ divides $\dim H$.

Lemma 3.1. *The order of $G(H)$ can not be q.*

Proof. Suppose on the contrary that $|G(H)| = q$. Let χ be an irreducible character of degree 2. By Lemma 2.1 (1) and the fact that $G[\chi]$ is a subgroup of $G(H)$, we know that $G[\chi] = \{1\}$ is trivial. It follows that the decomposition of $\chi\chi^*$ as (2.1) gives rise to a contradiction, since $4 < q$. Therefore, $a = 0$ and equation (3.1) is $2q^3 = q + q^2 b + 4q^2 c$, which is impossible. □

Lemma 3.2. *If $|G(H)| = q^2$ then $a = 0$ and $b \neq 0$. If $|G(H)| = q^3$ then H is of type $(1, q^3; q, q)$ as a coalgebra.*

Proof. If $|G(H)| = q^2$ then a similar argument as in Lemma 3.1 shows that $a = 0$. Therefore, equation (3.1) is $2q^3 = q^2 + q^2 b + 4q^2 c$. Obviously, $b \neq 0$, otherwise a contradiction will occur.

If $|G(H)| = q^3$ then Lemma 2.1 (2) shows that $a = c = 0$. □

Lemma 3.3. *If 2 divides both $|G(H)|$ and $|G(H^*)|$ then $H = R\#kC_2$ is a biproduct. Further, if kC_2 is normal in H then*

(1) H is self-dual, or

(2) H fits into an abelian central extension

$$k \to kC_2 \to H \to R \to k,$$

and $\dim V \le 2$ *for all irreducible H-comodule V.*

Proof. The first assertion follows from Lemma 2.10.

Since $R \cong H/H(kC_2)^+$ as a coalgebra and kC_2 is normal in H, we have that R is a Hopf algebra. If R is a dual group algebra then $R^* \subseteq G(H^*)$. It is impossible since q^3 does not divide $|G(H^*)|$. If R is a group algebra then $R^* \subseteq H^*$ is commutative and the index $[H^* : R^*] = 2$. Then it follows from the Frobenius Reciprocity [1, Corollary 3.9] that $\dim V \le 2$ for all irreducible H^*-module V. In this case, H fits into an abelian central extension as above by [20, Proposition 4.6.1]. Finally, if R is not trivial then it is self-dual [14]. Hence, $H^* \cong R^* \# k^{C_2} \cong R \# kC_2 = H$. □

Lemma 3.4. *If $|G(H)| = q^2$ or q^3 and H^* has a Hopf subalgebra K of dimension $2q^2$ then H fits into an extension*

$$k \to k\langle g\rangle \to H \to K^* \to k,$$

where $g \in G(H)$ is of order q. Further, if K is commutative then the extension is abelian.

Proof. Considering the map $\pi : H \to K^*$ obtained by transposing the inclusion $K \subseteq H^*$, we have that $\dim H^{co\pi} = q$. Notice that the dimension of every left coideal of H is $1, q$ or $2q$. Therefore, by Lemma 2.3, as a left coideal of H, $H^{co\pi}$ decomposes in the form $H^{co\pi} = k\langle g\rangle$, where $g \in G(H)$ is of order q. Thus $H^{co\pi}$ is normal in H, and hence H fits into an extension. The second assertion is obvious. □

Lemma 3.5. *If $|G(H)| = 2$ then*

(1) H is commutative, or

(2) H contains a Hopf subalgebra $K \subseteq H$ such that $K \cong k^F$, where F is a non-abelian group of order $2q^2$. Furthermore, the dimension of an irreducible H-module is at most q.

Proof. Observe that $a \neq 0$ in this case, since otherwise equation (3.1) can not hold. It follows that $G(H) \cup X_2$ spans a standard subalgebra of $R(H^*)$, which corresponds to a non-cocommutative Hopf subalgebra K of dimension $2+4a$. Then $2 + 4a = 2q, 2q^2$ or $2q^3$ by Nichols-Zoeller Theorem [23]. Obviously, $2 + 4a \neq 2q$ since otherwise equation (3.1) can not hold.

If $2 + 4a = 2q^3$ then H is of type $(1, 2; 2, \frac{q^3-1}{2})$ as a coalgebra. Then H is commutative by [2, Proposition 6.8].

If $2 + 4a = 2q^2$ then K is of type $(1, 2; 2, \frac{q^2-1}{2})$ as a coalgebra. In particular, $|G(H)| = 2$. By [21, Theorem 3.12.4], there are two isomorphism classes of non-trivial semisimple Hopf algebra of dimension $2q^2$, which are dual to each another:$\mathcal{B}_0$ and $\mathcal{B}_1 = \mathcal{B}_0^*$. In particular, $|G(\mathcal{B}_0)| = q^2$ and $|G(\mathcal{B}_1)| = 2q$. Therefore, K is trivial and hence commutative.

Finally, the Frobenius Reciprocity [1, Corollary 3.9] shows that $\dim V \le$ for all irreducible H-module V. □

Lemma 3.6. *If H contains a Hopf subalgebra $K \subseteq H$ such that $K \cong k^F$ and $G(H) \subseteq K$, where F is a non-abelian group of order $2q^2$. Then $G(D(H)^*)$ can not contain elements like $g \otimes \eta$, where $g \in G(H)$ and $\eta \in G(H^*)$ are of order 2.*

Proof. Suppose on the contrary that there is $g \otimes \eta \in G(D(H)^*)$ such that $g \in G(H)$ and $\eta \in G(H^*)$ are of order 2. Equivalently, there exists a non-trivial one-dimensional Yetter-Drinfeld module of H of the form $V_{g,\eta}$. See Lemma 2.9.

Consider the projection $\pi : H \to k^{\langle \eta \rangle}$ obtained by transposing the inclusion $k\langle \eta \rangle \subseteq H^*$. Since K is commutative and $G(H) \subseteq K$, $g^{-1}ag = a$ for all $a \in K^{co\pi}$. By [20, Theorem 1.6.4], $K^{co\pi}$ is a Hopf subalgebra of K. On the other hand, $\dim K^{co\pi} = 2q^2$ or q^2 by [20, Lemma 1.3.4] since $\dim \pi(K) = 1$ or 2. But the first possibility can not hold since K is not contained in $H^{co\pi}$ by Lemma 2.3. Hence, $\dim K^{co\pi} = q^2$ and $K^{co\pi}$ is a cocommutative Hopf subalgebra of H [13]. It is impossible since the order of $G(H)$ is 2 or $2q$. □

Remark 3.7. If $|G(H)| = 2$ and $|G(H^*)| = 2, 2q$ or $2q^2$ then [4, Lemma 2.2], Lemma 3.5, 3.6 and the proof of Corollary 2.7 imply that H or H^* must contain a non-trivial central group-like element. Hence, such Hopf algebra is not simple.

Lemma 3.8. *If $|G(H)| = 2q$ then $G(H)$ is cyclic.*

Proof. Notice that $X_2 \neq \varnothing$ and q^2 can not divide a. In addition, $|G[\chi]| = 2$ for all $\chi \in X_2$. Then $G(H)$ is abelian by [20, Proposition 1.2.6]. The lemma then follows from the classification of group of order $2q$ [3]. □

Proposition 3.9. *If $|G(H)| = 2$ then*
(1) $|G(H^*)| \neq 2$ and $2q$.
(2) If $|G(H^*)| = q^2$ or q^3 then H fits into an abelian extension

$$k \to k^F \to H \to k\langle g \rangle \to k,$$

where F is a group of order $2q^2$ and $g \in G(H^*)$ is of order q.
(3) If $|G(H^*)| = 2q^2$ then H fits into an abelian extension

$$k \to k^F \to H \to k\langle g \rangle \to k,$$

where F is a group of order $q^3, 2q^2$ or $2q$, and $g \in G(H^*)$ is of order $2, q$ or q^2.

Proof. Since the aim of our work is to classify non-trivial semisimple Hopf algebra, we may assume that H contains a commutative Hopf algebra $k^F \subseteq H$ of dimension $2q^2$ as in Lemma 3.5 and H is of type $(1, 2; 2, \frac{q^2-1}{2}; q, b; 2q, c)$ as a coalgebra, where $b + c \neq 0$.

(1) Suppose on the contrary that $|G(H^*)| = 2$. By Remark 3.7, H or H^* has a non-trivial central group-like element. Since $G(H^*) \cong G(H)$ as groups, we may assume that H contains such an element. We then consider the quotient Hopf algebra $\overline{H} = H/H(kG(H))^+$.

First, $\overline{H}$ is not trivial. Indeed, if $\overline{H}$ is a group algebra then $(\overline{H})^* \subseteq H^*$ is commutative. Thus $\dim V \leq [H^* : (\overline{H})^*] = 2$ for all irreducible H^*-module V [1, Corollary 3.9]. It is impossible. In addition, if $\overline{H}$ is a dual group algebra then $(\overline{H})^* \subseteq G(H^*)$. It is also impossible.

Second, $\overline{H}$ is not one constructed in [14]. Indeed, since the modules of $\overline{H}$ are those modules V of H such that each $x \in kG(H)$ acts as $\varepsilon(x)id_V$ on V, the number of one-dimensional modules of $\overline{H}$ is at most 2. Therefore, H is not the one constructed in [14]. All these facts imply that $|G(H^*)| \neq 2$.

Suppose on the contrary that $|G(H^*)| = 2q$. By Remark 3.7, H or H^* has a non-trivial central group-like element. If such element is of order 2 then H or H^* is self-dual by Lemma 3.3. This contradicts with the fact that $|G(H)| \neq |G(H^*)|$.

Let $g \in G(H), \eta \in G(H^*)$ and $\alpha \in G(H^*)$ be the element of order $2, 2$ and q, respectively. Then elements of $G(D(H)^*)$ are of the form $g \otimes \varepsilon$, $1 \otimes \eta$, $g \otimes \alpha^i$, $g \otimes \eta$ or $1 \otimes \alpha^i$ $(i = 1, \cdots, q)$. The discussion above, Lemma 3.6 and the proof of Corollary 2.7 imply that the first four possibilities can not exist. Therefore, the order of $G(D(H)^*)$ is q, which contradicts Lemma 2.8.

(2) By Lemma 3.4 and 3.5, H^* fits into an abelian extension

$$k \to k\langle g \rangle \to H^* \to kF \to k,$$

where F is a group of order $2q^2$ and $g \in G(H^*)$ is of order q. The result then follows after dualizing this extension.

(3) If $X_2 \neq \varnothing$ then $G(H^*) \cup X_2$ spans a standard subalgebra of $R(H)$, which corresponds to a quotient Hopf algebra L of dimension $2q^2 + 4a$. By Nichols-Zoelle Theorem, $\dim L = 2q^3$. Therefore, H is of type $(1, 2q^2; 2, \frac{q^3-q^2}{2})$ as an algebra. Then H or H^* has a non-trivial central group-like element [2, Theorem 6.4]. By Lemma 3.3, we may assume that H can not contain such an element since H is not self-dual. We shall examine every possible order of such an element g in $G(H^*)$.

If the order of g is 2 then $H^* = R\#k\langle g \rangle$ is a biproduct. It follows from Lemma 3.3 that H^* fits into an abelian extension

$$k \to k\langle g \rangle \to H^* \to R \to k.$$

If the order of g is q we then consider the extension

$$k \to k\langle g \rangle \to H^* \to H^*/H^*(k\langle g \rangle)^+ = \overline{H^*} \to k.$$

Since the number of one-dimensional modules of $\overline{H^*}$ is at most 2, $\overline{H^*}$ is trivial by [21, Theorem 3.12.4]. Moreover, $\overline{H^*}$ is not a dual group algebra since $|G(H)| = 2$. It follows that $\overline{H^*}$ is a group algebra and the extension above is abelian.

If the order of g is q^2 we then consider the extension

$$k \to k\langle g \rangle \to H^* \to H^*/H^*(k\langle g \rangle)^+ = \overline{H^*} \to k.$$

The classification of semisimple Hopf algebra of dimension $2q$ shows that $\overline{H^*}$ is trivial [5]. It is clearly that $\overline{H^*}$ is not a dual group algebra since $|G(H)| = 2$. Hence, $\overline{H^*}$ is a group algebra and the extension is abelian.

The result then follows after dualizing all these extensions above.

If $X_2 = \varnothing$ we then consider the projection $\pi : H^* \to kF$ obtained by transposing the inclusion $k^F \subseteq H$. Clearly, $\dim(H^*)^{co\pi} = q$. The decomposition of $(H^*)^{co\pi}$, as a coideal of H^*, shows that $(H^*)^{co\pi} = k\langle g \rangle$ is a group algebra, where $g \in G(H^*)$ is of order q. Therefore H fits into an abelian extension as described □

A semisimple Hopf algebra A is called group theoretical if the category of finite dimensional A-modules Rep$A$ is a group theoretical fusion category. As an immediate consequence of Proposition 3.9 and [18, Theorem 1.3.], we have the following result.

Corollary 3.10. *If $|G(H)| = 2$ then H is group theoretical.*

Lemma 3.11. *If $|G(H)| = 2q$ then*

(1) H is of type $(1, 2q; 2, \frac{q^3-q}{2})$ as a coalgebra, and H or H^* contains a non-trivial central group-like element, or

(2) H^* contains a normal Hopf subalgebra of dimension q, or

(3) H contains a Hopf subalgebra $K \subseteq H$ such that $K \cong k^F$, where F is a non-abelian group of order $2q^2$. Furthermore, the dimension of an irreducible H-module is at most q.

Proof. Observe that $X_2 \neq \varnothing$ and there is a non-cocommutative Hopf algebra of dimension $2q+4a$, corresponding to the standard subalgebra of $R(H^*)$ spanned by $G(H) \cup X_2$.

If $2q+4a = 2q^3$ then $H = K$ is of type $(1, 2q; 2, \frac{q^3-q}{2})$ as a coalgebra. Hence, H or H^* contains a non-trivial central group-like element [2, Theorem 6.4].

If $2q+4a = 2q^2$ then K is of type $(1, 2q; 2, a)$ as a coalgebra. By Lemma 3.4, if $|G(H^*)| = q^2$ or q^3 then H^* contains a normal Hopf subalgebra of dimension q. In all other cases, $H = H^{co\pi} \# kC_2$ is a biproduct by Lemma 3.3, where $\pi : H \to kC_2$ is a projection. Since K is not contained in $H^{co\pi}$ Lemma 2.3, $\dim K^{co\pi} = \dim(K \cap H^{co\pi}) \neq 2q^2$. Hence, $\dim \pi(K) = 2$ by [20, Lemma 1.3.4]. Therefore, $\pi_K : K \to kC_2$ is a surjection. Moreover, kC_2 is contained in K. Therefore, $K = K^{co\pi} \# kC_2$ is also a biproduct. By [27, Proposition 1.6], $K^* = (K^{co\pi})^* \# (kC_2)^* \cong (K^{co\pi})^* \# kC_2$ as a Hopf algebra. Furthermore, the description of group-like elements of a biproduct [24, 2.11] shows that the order of $G(K^*)$ is divisible by 2. All these facts imply that, comparing [21, Theorem 3.12.4], K is trivial and hence commutative.

Finally, the Frobenius Reciprocity [1, Corollary 3.9] shows that $\dim V \leq q$ for all irreducible H-module V. □

Proposition 3.12. *If $|G(H)| = q^3$ then*

(1) $kG(H)$ is a normal Hopf subalgebra of H and H^* has a central group-like element of order 2.

(2) The order of $G(H^*)$ can not be q^2 and q^3.

Proof. Since the index $[H : kG(H)] = 2$ is the smallest prime number dividing $\dim H$, the result in [10] shows that $kG(H)$ is a normal Hopf algebra of H. Then the Proposition follows from Lemma 2.2. □

The following corollary directly follows from the proposition above.

Corollary 3.13. *If $|G(H)| = q^3$ and $G(H)$ is abelian then H is group theoretical.*

Let q be a odd prime number. There are five classes of finite groups of order q^3 up to isomorphism: C_{q^3}, $C_q \times C_{q^2}$, $C_q \times C_q \times C_q$, $(C_q \times C_q) \rtimes C_q$ and $C_{q^2} \rtimes C_q$, where $\times$ denotes direct product and $\rtimes$ denotes semidirect product. The following consequence proves that $G(H)$ can not be the first one.

Corollary 3.14. *If $|G(H)| = q^3$ then $G(H)$ is not cyclic and H has a Hopf subalgebra of dimension $2q^2$.*

Proof. The group $G(H)$ acts by left multiplication on the set X_q. The set X_q is a union of orbits which have length $1, q, q^2$ or q^3. Since $|X_q| = q$ and the order of stabilizer $G[\chi]$ is at most q^2 for all $\chi \in X_q$, there is only one orbit which has length q. That is, $G[\chi]$ is of order q^2 for all $\chi \in X_q$.

Let $\chi \in X_q$. It follows from the results in [15, Section 2] that the exponent of $G[\chi]$ divides $deg\chi$. Hence, if $G[\chi]$ is cyclic then q^2 divides q, a contradiction. Thus $G(H)$ is not cyclic.

Since $|X_q| = q$ is odd, there is an irreducible character χ of degree q which is self-dual. Hence, $\{\chi\} \cup G[\chi]$ spans a standard subalgebra of $R(H^*)$, which corresponds to a Hopf subalgebra of dimension $2q^2$. □

Proposition 3.15. *If $|G(H)| = 2q^2$ then H is not simple.*

Proof. If $X_2 \neq \varnothing$ then there is a non-cocommutative Hopf algebra K of dimension $2q^2+4a$, corresponding to the standard subalgebra of $R(H^*)$ spanned by $G(H)\cup X_2$. By Nichols-Zoelle Theorem, $\dim K = 2q^3$. Therefore, $H = K$ and H is of type $(1, 2q^2; 2, \frac{q^3-q^2}{2})$ as a coalgebra. Then H or H^* contains a non-trivial central group-like element [2, Theorem 6.4].

If $X_2 = \varnothing$ we then consider the order of $G(H^*)$. By Lemma 3.4, if $|G(H^*)| = q^2$ or q^3 then H is not simple. In all other cases, it suffices to consider the case that H^* has a Hopf subalgebra K of dimension $2q^2$. Considering the map $\pi : H \to K^*$ obtained by transposing the inclusion $K \subseteq H^*$, we have $\dim H^{co\pi} = q$. Since the dimension of every irreducible left coideal of H is $1, q$ or $2q$, $H^{co\pi}$ decomposes in the form $H^{co\pi} = k\langle g\rangle$, where $g \in G(H)$ is of order q. Hence, $H^{co\pi}$ is normal Hopf subalgebra of H. □

Proposition 3.16. *If $|G(H)| = q^2$ then H is not simple.*

Proof. By Proposition 3.12, we may assume that $|G(H^*)| \neq q^3$. If $|G(H^*)| = 2, 2q$ or $2q^2$ then the proposition follows from Proposition 3.9, Proposition 3.15 and Lemma 3.11.

Finally, we consider the case that $|G(H^*)| = q^2$. Considering the map $\pi : H \to (kG(H^*))^*$ obtained by transposing the inclusion $kG(H^*) \subseteq H^*$, we have $\dim H^{co\pi} = 2q$. Notice that the dimension of every irreducible left coideal of H is $1, q$ or $2q$. Therefore, by Lemma 2.3, as a left coideal of H, $H^{co\pi}$ decomposes in the form $H^{co\pi} = k\langle g\rangle \oplus V$, where $g \in G(H)$ is of order q, and V is an irreducible left coideal of H of dimension q. Since gV and Vg are irreducible left coideals of H isomorphic to V, and gV, Vg are contained in $H^{co\pi}$, we have $gV = V = Vg$. Then [20, Corollary 3.5.2] shows that $k\langle g\rangle$ is a normal Hopf subalgebra of $k[C]$, where C is the simple subcoalgebra of H containing V, and $k[C]$ is a Hopf subalgebra of H generated by C as an algebra. Clearly, $\dim k[C] \geq q + q^2$. Moreover, by Nichols-Zoeller Theorem [23], $\dim k[C] = 2q^3, q^3$ or $2q^2$. If $\dim k[C] = 2q^3$ then $k[C] = H$ and $k\langle g\rangle$ is a normal Hopf subalgebra of H. If $\dim k[C] = q^3$ then the result follows from [10]. If $\dim k[C] = 2q^2$ then the result follows from Lemma 3.4. □

We shall afford two quite different proofs of the following proposition. The first one was pointed to the authors by professor S. Natale.

Proposition 3.17. *If $|G(H)| = 2q$ then H is not simple.*

First proof. By the discussion above, it suffices to consider the case that $|G(H^*)| = 2q$, and H and H^* both contain commutative Hopf subalgebras of dimension $2q^2$. Notice that, in this case, H is of type $(1, 2q; 2, \frac{q^2-q}{2}; q, 2q-2)$ as a (co)algebra.

Let $k^G \subseteq H$ and $k^F \subseteq H^*$ be the commutative Hopf subalgebras of dimension $2q^2$ obtained in Lemma 3.11. Considering the projection $\pi : H \to kF$ obtained by transposing the inclusion $k^F \subseteq H^*$, we have $\dim H^{co\pi} = q$. There are two possible decompositions of $H^{co\pi}$ as a left coideal of H:

(1) $H^{co\pi} = k\langle g\rangle$, where $g \in G(H)$ is of order q.

(2) $H^{co\pi} = k1 \oplus \sum_i V_i$, where V_i is an irreducible left coideal of H of dimension 2.

If the first one holds true then H is not simple, and H also fits into an abelian extension.

If the second one holds true then we may assume $H^{co\pi} \subseteq k^G$. Notice that the number of irreducible coideals of dimension 2 in $H^{co\pi}$ is $\frac{q-1}{2}$.

Note that $H^{co\pi}$ is a Yetter-Drinfeld submodule of H, and decompose it as a sum of irreducible Yetter-Drinfeld submodules. We may assume that there is a unique summand (spanned by 1) of dimension 1 (otherwise H would contain a central group-like element, and we are done).

Now, the dimension of every such irreducible summand W must divide the dimension of H. On the other hand, it must be of the form $2n$, where $1 \leq n \leq \frac{q-1}{2}$ (this follows after decomposing W, which is a coideal, into a sum of irreducible coideals). So that $n = 1$, because $n = q^k > \frac{q-1}{2}$ if $k > 1$.

But this implies that the V_i's are Yetter-Drinfeld submodules of H. In particular, $D(H)$ has irreducible modules of dimension 2 and thus $G(D(H)^*)$ has an element $g \times \eta$ of order 2, since $D(H)$ does not has irreducible modules of dimension 3. □

Second proof. If the second one holds true then $H^{co\pi} \subseteq K$ and $K^{co\pi} = H^{co\pi}$, where we write $K = k^G$. Then $\dim K = \dim K^{co\pi} \dim \pi(K)$ [20, Lemma 1.3.4] implies that $\dim \pi(K) = 2q$. Consider the surjection $\pi|_K : K \to \pi(K)$. Since $kG(K) \cap K^{co\pi} = k1$, we have that $\pi|_{kG(K)} : kG(K) \to \pi(K)$ is an isomorphism. Hence $K = K^{co\pi} \# kG(K)$ is a biproduct by the Radford's projection theorem [24, Theorem 3]. Since K is commutative, $kG(K)$ is normal in K. It follows that $K^{co\pi}$ is an ordinary Hopf algebra, since $K^{co\pi} \cong K/K(kG(K))^+$ as a coalgebra. Since $G(K)$ is cyclic and $K^{co\pi}$ is a group algebra of a cyclic group of order q, we know that $kG(K)$ and $K^{co\pi}$ are both self-dual. Therefore, $K^* \cong (K^{co\pi})^* \# k^{G(K)} \cong K^{co\pi} \# kG(K) = K$. But this contradicts the fact that the finite group G is non-abelian. □

Up to now, we have examined every possible order of $G(H)$ and proved that H is not simple in all cases. Therefore, together with Proposition 2.4 and the Burnside's $p^a q^b$-Theorem, we obtain our main theorem.

Theorem 3.18. *Let A be a semisimple Hopf algebra of dimension $2q^3$, where q is a prime number. Then A is semisolvable.*

Remark 3.19. The theorem above implies that the analogue of Burnside's $p^a q^b$-Theorem for semisimple Hopf algebras of dimension $2q^3$ is true. However, there exists a semisimple Hopf algebra of dimension $p^2 q^2$ which is simple as Hopf algebra [8]. Therefore the analogue of Burnside's $p^a q^b$-Theorem does not always hold for semisimple Hopf algebras.

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